Lattice Quantum Chromodynamics

- Wilson’s Formulation of Lattice-QCD
  - Including gauge fields
- A first application
  - Static potential
Lattice QCD

change:

- 2-d $\rightarrow$ 4-d
- gauge field $U(x, \mu) \in U(1) \rightarrow U(x, \mu) \in SU(3)$
- Pauli matrices $\sigma_\mu \rightarrow$ Gellmann-matrices $\gamma_\mu$
- spinors become 12-component complex vectors
- theory needs renormalization
The Lattice Gauge Action

Euclidean pure gauge action

\[ S = a^4 \sum_x \sum_{\mu,\nu}^{\mu < \nu} \frac{N}{g_0^2} \text{Tr} \left[ 2 - (U_p + U_p^\dagger) \right] \equiv \beta \]

\[ U_p = U(x, \mu)U(x + \mu, \nu)U^{-1}(x + \nu, \mu)U^{-1}(x, \nu) \]

where \( g_0 \) is the bare gauge coupling \( N \) relates to the group

\( N = 1 \) for U(1), \( N \) for SU(N), for QCD: \( N = 3 \)

the action converges in the continuum limit \( a \to 0 \) to the expression

\[ S \to \frac{N}{g_0^2} F_{\mu\nu}F_{\mu\nu}, \quad a \to 0 \]

note that \( 2 - U_p - U_p^\dagger = (1 - U_p^\dagger)(1 - U_p) \geq 0 \)
the path integral is then given

\[ Z = \int DU(x, \mu) e^{-S} \]

where the measure is evaluated by a group parametrization

for \( U(1), U(x, \mu) \in U(1) \) we have \( U(x, \mu) = e^{i\theta(x, \mu)} \)

\[ \int DU(x, \mu) = \prod_{x, \mu} \frac{1}{2\pi} d\theta(x, \mu) \]

the measure has the following properties (Haar measure)

\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta = 1 \quad \{ \int dU = 1 \} \]

\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta = 0 \quad \{ \int dUU = 0 \} \]

\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} d(\theta + \vartheta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \quad \{ \int d(gU h^{-1}) = \int dU \} \]
Physical Quantities = Gauge Invariant Observables

observables are given by

$$\langle O \rangle = \frac{1}{Z} \int D U(x, \mu) O(U) e^{-S}$$

$$= \frac{1}{Z} \int D g(x) \int D U(x, \mu) O(U) e^{-S}$$

$$\int d g(x) = 1$$

$$= \frac{1}{Z} \int D U_g(x, \mu) \int D g O(U_g) e^{-S}$$

$$= \frac{1}{Z} \int D U(x, \mu) \left[ \int D g O(U_g) \right] e^{-S}$$

$$\int D g O(U_g)$$ is only non-zero if $O(U_g)$ is gauge invariant

- only consider gauge invariant observables, no gauge fixing (Elitzur’s theorem)
- explicit form of the measure is only needed in perturbation theory
A first application: The static potential → confinement

start with 2-point-function

\[ G(q_N, t_N; q_0, t_0) = \langle q_N | e^{-\frac{i}{\hbar}H(t_N-t_0)} | q_0 \rangle, \quad H = \frac{\hat{p}^2}{2m} + V(\hat{q}) \]

\[ \Rightarrow \lim_{m \to \infty} G(q_N, t_N; q_0, t_0) = \delta(q_N - q_0) e^{-V(q)(t_N-t_0)} \]

→ in the infinite mass limit we obtain the static potential

\[ V(q) = -\lim_{t \to \infty} \left( \frac{1}{t} \ln \left[ \lim_{m \to \infty} G(q, t; 0, 0) \right] \right) \]

note: order of limits is important
(what do we obtain when the limits are interchanged?)
a gauge invariant 2-point function

with $|x - y| \equiv R$

$$G(t, x, y) \equiv \langle \bar{\psi}(t, x) \prod_{i}^{R-1} U(t, x + i, i) \psi(t, y) \rangle$$

for $t \gg 1$

$$[G(0, x, y)G(T, x, y)] \propto e^{-Et}$$

for $m \to \infty$ the energy $E$ gives the potential energy of static charges

for $m \to \infty$ the matter fields become static

→ gauge variant object
a *gauge invariant* observable

→ gauge invariant time propagation

\[
\lim_{T \to \infty} \left[ -\frac{1}{T} \ln \langle W(C) \rangle \right] = V(R)
\]

→ potential between two static charges
An evaluation of the Wilson Loop

\[ \langle W(C) \rangle = \frac{1}{Z} \int \mathcal{D}UW(C) e^{-\beta a^4 \text{Tr} \sum_p (U_p + U_p^\dagger)} \]

expansion in \( \beta \ll 1, \quad \beta = \frac{N}{g_0^2} \) \textit{strong coupling}:

\[ \langle W(C) \rangle \approx \int \mathcal{D}UW(C) \prod_p (1 - \beta (U_p + U_p^\dagger)) + \ldots \]

\[ \approx \int \mathcal{D}UW(C) (1 - \beta \sum_p (U_p + U_p^\dagger)) + \ldots \]
• lowest order: $U$'s appear isolated $\Rightarrow \int DUU = 0$

• order $\beta$:

again isolated $U$'s $\Rightarrow$ no contribution

• first non-vanishing contribution: complete coverage of the area of the loop

$\Rightarrow \int DU1_{\beta^{TR}} = \beta^{TR}$
$$\langle W(C) \rangle \propto \beta^{RT} = e^{RT \ln \beta}$$

potential:

$$V(R) = -\frac{1}{T} \ln \langle W(C) \rangle \propto -\ln \beta R \equiv \sigma R$$

this gives a linearly increasing potential

→ (linear) confinement of static charges with

string tension $\sigma$

- we only used link variables and properties of the Haar measure
- first success of lattice gauge theory
The full picture

- for small coupling $g_0 \ll 1 \Leftrightarrow \beta \gg 1$ the Wilson loop maybe evaluated perturbatively

$\rightarrow$ Coulomb piece $V(R) = \frac{\alpha(\mu)}{R}$, $\alpha = \frac{g^2}{4\pi}$

$\rightarrow$ $\alpha(\mu)$ running coupling constant determined at scale $\mu$

$\rightarrow$ scale dependence of $\alpha$ on $\mu \leftrightarrow$ asymptotic freedom

$\Rightarrow$ Ianus-head of QCD: same theory describes

<table>
<thead>
<tr>
<th>confinement</th>
<th>asymptotic freedom</th>
</tr>
</thead>
<tbody>
<tr>
<td>spectrum of bound states</td>
<td>system of</td>
</tr>
<tr>
<td>hadrons</td>
<td>weakly interacting</td>
</tr>
<tr>
<td>glueballs</td>
<td>quarks and gluons</td>
</tr>
</tbody>
</table>
string picture

at large distances the quarks are pulled apart and a string is building up

the excitations of this string gives the string spectrum

$\rightarrow$ reveals another piece proportional $-\frac{\pi}{12} \cdot \frac{1}{R}, \quad R \gg 1$
Static potential in the pure gauge theory on the lattice

\[ V(R) = \frac{\alpha}{R} - \frac{\pi}{12} \cdot \frac{1}{R} + \sigma R \]

lattice simulation in SU(3) lattice gauge theory
Testing The Force between static quarks

- **Coulomb piece**
  \[ F(r) = \frac{\alpha}{r^2} \]
  \( \alpha \) coupling

- **Confinement piece**
  \[ F(r) = \sigma \]
  \( \sigma \) string tension

- **string piece**
  \[ F(r) = -\frac{\pi}{12} \cdot \frac{1}{r} \]
  string picture

![Diagram](image.png)
Test of different pieces

- short distance part (Necco and Sommer)
  \[ r_0 = 0.5 \text{fm} \]

- long distance part (Lüscher and Weisz)
Including matter fields

rescaling $\Psi \rightarrow \sqrt{\kappa} \Psi$, $\kappa = 1/(2m_q + 8)$

\[
S = a^4 \sum_x \left\{ \overline{\Psi}(x) \Psi(x) - \kappa \overline{\Psi}(x) \sum_\mu \left[ (1 - \gamma_\mu) U^\dagger(x - \mu, \mu) \Psi^L(x - a\mu) + (1 + \gamma_\mu) U(x, \mu) \Psi(x + a\mu) \right] \right\}
\]

\[
\equiv a^4 \sum_{xy} \overline{\Psi}_x D_{xy} \Psi_y
\]

\[
D_{xy} = \delta_{x,y} - \kappa \sum_\mu \left[ (1 - \gamma_\mu) U^\dagger(x - \mu, \mu) \delta_{x,x-\mu} + (1 + \gamma_\mu) U(x, \mu) \delta_{x,x+\mu} \right]
\]

integrating out the fermions

\[
Z = \int \mathcal{D}\overline{\Psi} \mathcal{D}\Psi \, e^{\overline{\Psi} D \Psi} = \det D = e^{-\text{Tr} (\text{spin, color, } x) \ln(D)}
\]
Effective action

\[ S_{\text{eff}}(U) = S_g(U) - \text{Tr} \ln D \equiv S_g(U) - \text{Tr} \ln(1 - \kappa M(U)) \]

for \( \kappa \) being small one can perform a hopping parameter expansion

\[ -\text{Tr} \ln(1 - \kappa M(U)) = \sum_l \frac{\kappa^l}{l} \text{Tr} M^l(U) \]

Wilson loop in the hopping parameter expansion

- set \( \beta = 0 \)

- take the smallest Wilson loop, i.e. the plaquette \( \langle U_p \rangle \)
\[ \langle U_p \rangle = \int \mathcal{D}U U_p e^{-\text{Tr} \ln (1 - \kappa M(U))} \approx \int \mathcal{D}U U_p [1 - \kappa \sum_{x, \mu} (U(x, \mu) + U(x, \mu)^\dagger) + \ldots] \]

- in lowest order in \( \kappa \) the integral vanishes
- in \( O(\kappa) \) one link of the plaquette can be compensated with a link from the effective action ... but only one
- it is only in order \( \kappa^4 \) where all links from the plaquette can be compensated by links from the effective action

In \( O(\kappa^4) \) we get terms of the form

\[
\text{Tr}_{(color)} \left[ U(x, \mu)^{-1} U(x + \mu, \nu)^{-1} U(x + \mu + \nu, \mu) U(x, \nu) \right] \cdot \text{Tr}_{(spin)} \left[ (1 - \gamma_\mu)(1 - \gamma_\nu)(1 + \gamma_\mu)(1 + \gamma_\nu) \right]
\]

which evaluate to \( -32\kappa^4 \sum_P \text{Tr} U_P^\dagger \)
• in lowest order \((\kappa^4)\) the effective action from the fermions just amounts in a shift \(\beta', \beta' = \beta - 32\kappa^4\)

disorder effect of fermions

• if we consider larger Wilson loops of size \(R \times T\) we find

for \(\beta \ll 1\) and \(\kappa \ll 1\): \(\langle W(C) \rangle \approx \beta^{RT} + \kappa^{2(R+T)}\)

Thus we find in addition to the area law a perimeter law

for very large Wilson loop, the perimeter law will dominate
and the potential will go to a constant, the screening energy

\[ V(R) = \lim_{T \to \infty} \left[-\frac{1}{T(2T + 2R)} \ln \kappa\right] = -2 \ln \kappa \]

when matter fields are included the quarks are no longer confined!

we speak of a breaking of the string \(\rightarrow\) hadronization
String breaking

- dynamical quarks -
String breaking
(first results from SESAM Collaboration)

at a distance of $R \approx 1\text{fm}$

- ground state change from potential energy $V(R)$ of a static $\bar{q}q$ state
- to the energy of 2 mesons state $\bar{B}B$
Summary of lecture II

• Wilson’s formulation of lattice QCD:

\[ S = \sum_x \sum_{\mu,\nu}^{\mu<\nu} \beta \text{Re} \text{Tr} (U_p) + \sum_x \bar{\Psi}(x) \left[ m_q + \gamma_\mu (\nabla_\mu + \nabla^*_\mu) - r \nabla^*_\mu \nabla_\mu \right] \Psi(x) \]

• static potential for pure gauge field theory

\[ V(R) = \frac{\alpha}{R} - \frac{\pi}{12} \cdot \frac{1}{R} + \frac{\sigma R}{\text{Coulomb}} - \frac{\pi}{12} \cdot \frac{1}{R} + \frac{\sigma R}{\text{String}} + \frac{\sigma R}{\text{Confimenent}} \]

• String breaking in presence of dynamical quarks